arXiv:1007.0861v1 [math.CO] 6 Jul 2010

Deformed Kazhdan-Lusztig elements and Macdonald polynomials

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Abstract

We introduce deformations of Kazhdan-Lusztig elements and degenerate nonsymmetric Macdonald polynomials, both of which form a distinguished basis of the polynomial representation of the maximal parabolic subalgebra of the Hecke algebra. We give explicit integral formula for these polynomials, and explicitly describe the transition matrices between classes of polynomials. We further develop a combinatorial interpretation of homogeneous evaluations using an expansion in terms of Schubert polynomials in the deformation parameters.

1. Introduction

We discuss two important bases in the Iwahori-Hecke algebra $\mathcal{H}$: the Kazhdan-Lusztig basis [1] and Young's seminormal, or orthogonal basis [2, 3]. While the former admits a deep geometric interpretation, the latter is purely algebraic and combinatorial. We study these two bases for the maximal parabolic case in a very explicit way using the polynomial representation of $\mathcal{H}$. In this representation, the parabolic KL basis [4] gives rise to KL elements, while the orthogonal basis elements are given by specialised non-symmetric Macdonald polynomials [5, 6, 7]. We review their construction and for both cases give elegant explicit formulas using factorisations in terms of Baxterised operators. We further introduce natural deformations which interpolate bewteen classes of KL elements and specialised Macdonald polynomials, as well as provide explicit multi-dimensional integral expressions for the polynomial basis elements.

Distinguished bases in finite dimensional polynomial representation modules of the Hecke algebra have important applications in physics. For example, they are related to quantum incompressible states [8], which in the limit to Jack polynomials contain model groundstate wave functions of (fractional) quantum Hall systems [9]. They are also used to describe polynomial solutions to the q-deformed Knizhnik-Zamolodchikov (qKZ) equation [10, 11, 12]. The latter are, for example, relevant for critical percolation in an inhomogeneous system [13, 14, 15, 16, 17, 18], as well as to the spin quantum Hall transition [19].

There is a rich combinatorial content in this theory, and we proceed to derive combinatorial rules for certain transition matrices between basis elements. In addition we consider homogeneous evaluations which give rise to enumeration formulas for combinatorial objects such as fully packed loop configurations, alternating sign matrices and punctured symmetric plane partitions, see e.g. [20]. Using the theory of Schubert polynomials, we obtain natural combinatorial interpretations of the deformation parameters.

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2. Polynomial representation of the Hecke algebra

The Hecke algebra $\mathcal{H}$ of the symmetric group $W = S_n$ generated by the simple reflections s_i, is the algebra over $\mathbb{Q}[t, t^{-1}]$ defined in terms of generators $T_i \equiv T_{s_i}$, $i = 1, \ldots, n-1$, and relations

$$\begin{aligned} &(T_i - t)(T_i + t^{-1}) = 0, \qquad T_iT_j = T_jT_i \quad \forall\, i, j : |i - j| > 1, \\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}. \end{aligned} \tag{1}$$

The Baxterised element $T_i(u) \in \mathbb{C}\mathcal{H}$ is defined by

$$T_i(u) = T_i + \frac{t^{-u}}{[u]_t},$$

where the notation $[u]_t$ stands for the usual t-number

$$[u]_t = \frac{t^u - t^{-u}}{t - t^{-1}}.$$

The notation $[u]$ will always refer to base t. The standard projectors are obtained by specialising u:

$$T_i(1) = T_i + t^{-1}, \qquad T_i(-1) = T_i - t.$$

The Hecke algebra has a multivariate polynomial representation, which is conveniently expressed in terms of the divided difference operator ∂_i, defined by

$$\partial_i f = \frac{f - s_i f}{z_i - z_{i+1}}, \tag{2}$$

where s_i denotes the transposition $z_i \leftrightarrow z_{i+1}$. The projector $T_i(1)$ then induces the operator ∇_i,

$$\nabla_i = (tz_i - t^{-1}z_{i+1})\partial_i := \frac{tz_i - t^{-1}z_{i+1}}{z_i - z_{i+1}}(1 - s_i), \tag{3}$$

which commutes with multiplication with functions symmetric in z_i, z_{i+1}, and acts on 1 and z_i as

$$\nabla_i : \begin{cases} z_i & \mapsto & tz_i - t^{-1}z_{i+1} \\ 1 & \mapsto & 0 \end{cases} \tag{4}$$

Representations of the Hecke algebra are labeled by partitions. In this paper we will restrict ourselves to spaces corresponding to rectangular partitions with two rows or two columns, and in which basis elements can be labeled by partitions contained in the maximal staircase partition. It is our aim to describe several distinguished polynomial bases.

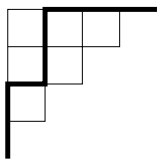
We will sometimes indicate partitions by their corresponding Yamanouchi word. Recall that a Yamanouchi word w is a word with integer entries such that for every factorisation $w = w'w''$, the right factor w'' contains as many or more occurrences of the symbol i than of $i + 1$, for all $i \geq 0$. Given any Yamanouchi word, a dual Yamanouchi word may may be obtained by the following procedure: Begin by numbering, from right to left, each integer of the Yamanouchi word that is equal to 0. Then repeat for each integer equal to 1, and so on up to the maximal integer appearing in the Yamanouchi word. The resultant label from this numbering is also a Yamanouchi word, which we call its dual.

Example 1. *We begin with the Yamanouchi word* 11011000*, and follow the procedure described above. For any i, label each occurence of i from right to left, by* $0, 1, 2, \ldots$*:*

1	1	0	1	1	0	0	0
		3			2	1	0
3	2		1	0			
3	2	3	1	0	2	1	0

This yields the Yamanouchi word 32310210 *dual to* 11011000*.*

There is a further simple bijection between Yamanouchi words with two distinct integers, and sub-partitions of a staircase partition. This is described as follows. Labeling each vertical step with a 1, and each horizontal step with a 0, the staircase partition is labeled $1010\ldots10$, which is the Yamanouchi word dual to the word $nn(n-1)(n-1)\ldots1100$. Any sub-partition of this staircase will have a lower edge which is a Dyck path, and its labeling in terms of 1s and 0s will form a Yamanouchi word, e.g. the Dyck path arising from $[1,1] \subset [3,2,1]$ is



$\rightarrow \quad 11011000.$

2.1. Kazhdan-Lusztig basis from t-Vandermonde determinant

Kazhdan and Lusztig (KL) [1] defined a linear basis of the Hecke algebra $\mathcal{H}$, in relation with a fundamental involution of $\mathcal{H}$. KL's construction provides a linear basis of any specific polynomial representation. In the space that we consider, this basis can be indexed by Yamanouchi words, and can be defined, up to normalisation, by vanishing properties. Let $v, w \in \mathbb{N}^{2n}$ be a Yamanouchi word, and let $t^{-2w} = \prod_i t^{-2w_i}$, then

$$\mathrm{KL}_v(z_1, \ldots, z_{2n})\Big|_{z=t^w} = 0 \quad \text{unless} \quad w = v. \tag{5}$$

KL basis elements can also be obtained as follows. First defining the t-Vandermonde Δ_t by

$$\Delta_t(z_1, \ldots, z_N) = \prod_{1 \le i < j \le N} (tz_i - t^{-1}z_j). \tag{6}$$

Then it was shown in [21, 22] that the KL basis can be generated by applying operators of the form $T_i(u)$ to $\Delta_t(z_1, \ldots, z_n)\Delta_t(z_{n+1}, \ldots, z_{2n})$, with appropriate arguments u. For example, for $n = 3$ we find:

$$\begin{aligned}
\mathrm{KL}_{111000} &= \Delta_t(z_1, z_2, z_3)\Delta_t(z_4, z_5, z_6),\\
\mathrm{KL}_{110100} &= T_3(1)\ \mathrm{KL}_{111000},\\
\mathrm{KL}_{101100} &= T_2(1)T_3(2)\ \mathrm{KL}_{111000},\\
\mathrm{KL}_{110010} &= T_4(1)T_3(2)\ \mathrm{KL}_{111000},\\
\mathrm{KL}_{101010} &= T_2(1)T_4(1)T_3(2)\ \mathrm{KL}_{111000}.
\end{aligned}$$

The order in which the Baxterised operators $T_i(u)$ are applied is determined by a Young diagram. For example, if we graphically denote $T_i(u)$ by a labeled tilted square

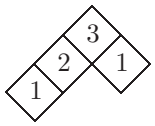
$T_i(u) =$

,

acting from top to bottom in the ith column, then the polynomial

$$\mathrm{KL}_{10111000} = T_5(1)T_2(1)T_3(2)T_4(3)\ \mathrm{KL}_{11110000},$$

may be graphically depicted as



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Rotating these pictures by $\pi/4$, it is clear that we may index the KL basis by labeled partitions, the index i of each operator $T_i(u)$ being determined by the position of the box.

$$
\begin{aligned}
&\mathrm{KL}_{111000} = \mathrm{KL}_{\emptyset} = \emptyset \quad \big(= \Delta_t(z_1,z_2,z_3)\Delta_t(z_4,z_5,z_6)\big),\\
&\mathrm{KL}_{110100} = \mathrm{KL}_{[1]} = T_3(1)\ \mathrm{KL}_{\emptyset} = \boxed{1}\ ,\\
&\mathrm{KL}_{101100} = \mathrm{KL}_{[1,1]} = T_2(1)\ T_3(2)\ \mathrm{KL}_{\emptyset} = \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}\ ,\\
&\mathrm{KL}_{110010} = \mathrm{KL}_{[2]} = \begin{array}{|c|c|}\hline 2 & 1\\ \hline\end{array}\\
&\mathrm{KL}_{101010} = \mathrm{KL}_{[2,1]} = \begin{array}{|c|c|}\hline 2 & 1\\ \hline 1 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\ .
\end{aligned}
$$

2.2. Young-Murphy basis from t-Vandermonde determinant

The non-symmetric vanishing Macdonald polynomial [23, 24] of index $n-1,\dots 0, n-1,\dots,0$ specializes for $q=t^6$ to the product of t-Vandermonde determinants [11], which is a homogeneous polynomial. The action of the Hecke algebra on this polynomial generates a space of homogeneous polynomials which is an irreducible module.

The Macdonald polynomials specialised at $q=t^6$, and indexed by Yamanouchi words which are a permutation of $n-1,\dots 0, n-1,\dots,0$, form a basis of this module. These polynomials retain the vanishing properties of the unspecialised ones. Like the KL polynomials, they may also be indexed by a labeled Young diagram corresponding to a partition $[\lambda]$, and we shall write $\mathrm{M}_{[\lambda]}$. The integers corresponding to the maximal element are

$$
\begin{array}{|c|c|c|c|c|c|}\hline
n & n{-}1 & \dots & \dots & 3 & 2\\ \hline
n{-}1 & n{-}2 & & & 2 & \\ \cline{1-5}
 & & & & & \\ \cline{1-4}
 & & & & & \\ \cline{1-3}
3 & 2 & & & & \\ \cline{1-2}
2 & & & & & \\ \cline{1-1}
\end{array}
$$

Example 2. *For $n=3$ the irreducible module corresponds to*

$$
\begin{aligned}
\mathrm{M}_{210210} &= \Delta_t(z_1,z_2,z_3)\Delta_t(z_4,z_5,z_6)\\
\mathrm{M}_{212010} &= T_3(3)\ \mathrm{M}_{210210}\\
\mathrm{M}_{221010} &= T_2(2)\ T_3(3)\ \mathrm{M}_{210210}\\
\mathrm{M}_{212100} &= T_4(2)\ T_3(3)\ \mathrm{M}_{210210}\\
\mathrm{M}_{221100} &= T_2(2)\ T_4(2)\ T_3(3)\ \mathrm{M}_{210210}
\end{aligned}
$$

and which we abbreviate by

$$
\mathrm{M}_{210210} = \emptyset, \qquad \mathrm{M}_{212010} = \boxed{3}\ ,
$$

$$
\mathrm{M}_{221010} = \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline\end{array}\ , \quad \mathrm{M}_{212100} = \begin{array}{|c|c|}\hline 3 & 2\\ \hline\end{array}\ , \quad \mathrm{M}_{221100} = \begin{array}{|c|c|}\hline 3 & 2\\ \hline 2 & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\ .
$$

Given the KL and M bases, it is natural to look at the transition matrices between them. For example, let $\mu = n-1, n-1,\dots,221100$ denote the maximal Yamanouchi word. Then $\tilde{\mu} = 1010\dots 10$ is the Yamanouchi word dual to μ, whose corresponding partition is the staircase

partition $[\rho] = [n-1, \ldots, 1]$. Then the expansion of the the maximal basis element M_μ in terms of KL polynomials was conjectured to be given by (equation (5.3) in [22])

$$M_\mu =: M_{[\rho]} = \sum_{[\lambda]\subseteq[\rho]} \tau^{-n_\lambda}\ \mathrm{KL}_{[\lambda]}, \qquad \tau = -[2]. \tag{7}$$

Here n_λ is defined as the signed sum of boxes between the Dyck path corresponding to the maximal staircase $[\rho]$, and the Dyck path corresponding to the partition $[\lambda]$. This is most easily seen via an example:

Example 3. *Consider $n = 6$, with $[\lambda] = [3, 1]$ as shown in Figure 1. In this example we this have $n_\lambda = 5 - 4 + 2 = 3$.*

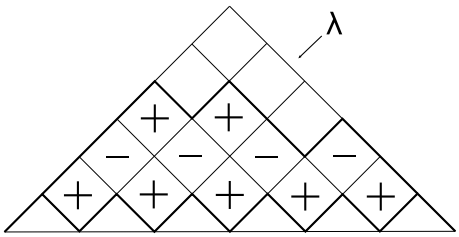


Figure 1: Definition of n_λ as the sum of signed boxes. The first row of boxes is labelled with plus signs, the next with minus signs, and so on.

We will prove a more general result than (7) in Theorem 2, see Section 3.2.

2.3. Evaluations

The evaluations $z_i = 1$ of certain degenerate Macdonald and KL polynomials, normalised by dividing by the t-Vandermonde determinants, correspond to the number of (punctured) totally symmetric self complementary plane partitions (PTSSCPPs) with a weight in $\tau = -t - 1/t$. Such enumerations were considered in [25, 14, 26, 27, 28] using methods which we will discuss in Section 3.1. Let us first give an example of two evaluations:

Example 4. *Let $\Delta = \Delta_t(z_1, z_2, z_3)\Delta_t(z_4, z_5, z_6)$. The ratio*

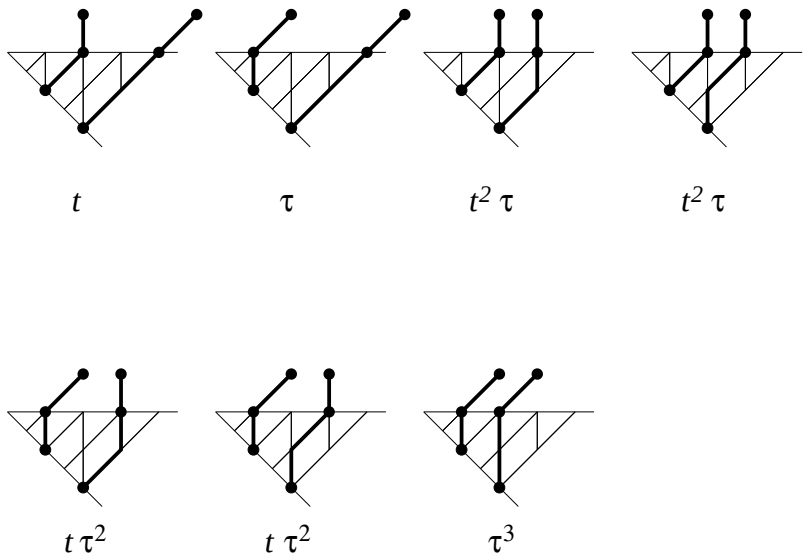
$$\frac{1}{\Delta} M_{221100}\Big|_{z_i=1} = 3\tau^{-1} + 3\tau + \tau^3. \tag{8}$$

A different result is obtained when specialising the sum of the KL basis, i.e. taking $c_\lambda = 0$ in (7).

$$\frac{1}{\Delta}\left(KL_{111000} + KL_{110100} + KL_{101100} + KL_{110010} + KL_{101010}\right)\Big|_{z_i=1} = 1 + 3\tau + 2\tau^2 + \tau^3. \tag{9}$$

TSSCPPs, see e.g. [29, 30, 20], are in simple bijection to nonintersecting lattice paths consisting of north and north-east steps, and starting at positions $(i, -i)$ and ending on the x-axis. We further augment these paths, following [14], with an extra step between the lines $y = 0$ and $y = 1$, in such a way that the difference of successive x coordinates of the end positions are odd numbers, and the coordinate of the first endpoint is 1. We also assign a weight a weight τ to vertical steps, and a weight t to vertical augmented step.

Example 5. *The seven weighted paths for $n = 2$ are*



The weighted enumeration of these augmented paths is $N(t,\tau) = \tau^3 + 2t\tau^2 + 2t^2\tau + \tau + t$.

Note that for special values of t we obtain the evaluations in Example 4.

$$N(\tau^{-1},\tau) = 3\tau^{-1} + 3\tau + \tau^3, \qquad N(1,\tau) = 1 + 3\tau + 2\tau^2 + \tau^3.$$

In the next section we clarify this correspondence and give a unified description of the M and KL polynomials.

3. Deformations

Taking $v \in \mathbb{N}^{2n}$, let $\langle v\rangle$ denote a *spectral vector* of length $2n$. From a starting polynomial M_{v_0} in $z_1,\dots,z_{2n+2}$, one generates other polynomials by the recursion

$$M_v \to M_{s_i v} := T_i\big(\langle v\rangle_{i+1} - \langle v\rangle_i\big)M_v,$$

requiring only that i be such that $v_i < v_{i+1}$, $\langle v\rangle_{i+1} \neq \langle v\rangle_i$ and $\langle s_i v\rangle = s_i\langle v\rangle$. Thanks to the Yang-Baxter equation, the polynomials M_w are independent of the choice of the reduced decomposition chosen to pass from v to w.

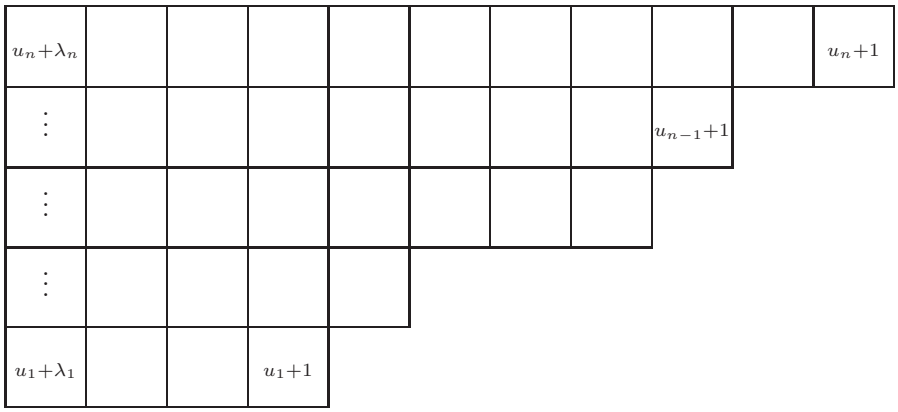
Let us now be more specific and take

$$v_0 = [n,\dots,0,n,\dots,0], \qquad \langle v_0\rangle = (n-1-u_1, n-2-u_2, \cdots, -u_n, n, n-1,\dots,2,1).$$

As we did in the case of KL polynomials, let us rather use partitions for indexing. The starting point is $M_\emptyset = M_{v_0} = \Delta_t(z_1,\dots,z_{n+1})\Delta_t(z_{n+2},\dots,z_{2n+2})$, then an arbitrary partition $\lambda =$

$[\lambda_n, \ldots, \lambda_1]$ with $\lambda_n \geq \lambda_{n-1} \geq \ldots \geq \lambda_1$ is filled as follows

$M_{[\lambda_n,\ldots,\lambda_1]}(u_1, \ldots, u_n) =$



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where we have suppressed the dependence on $z_1, \ldots, z_{2n+2}$. The usual Macdonald polynomials are obtained by setting $u_i = i - \lambda_i + 1$.

Example 6. *The deformed maximal Macdonald polynomial (for $q = t^6$) is given by*

$$\mathrm{M}_{[2,1]}(u_1, u_2) = \mathrm{M}_{221100}(u_1, u_2) = \begin{array}{|c|c|}\hline u_2+2 & u_2+1 \\ \hline u_1+1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = T_2(u_1+1)T_4(u_2+1)T_3(u_2+2)\ M_{210210}.$$

In the following we will interchangeably use the notation

$$\psi_{a_1\ldots a_n}(u_1, \ldots, u_n) = M_v(u_1, \ldots, u_n) = M_{[\lambda_n,\ldots,\lambda_1]}(u_1, \ldots, u_n),$$

with $a_k = \lambda_k + k$.

3.1. Integral formulas

Introducing an auxiliary variable w, we can separate the variables z_i and z_{i+1} and note the following

$$T_i(1)\frac{\Delta_t(z_i, z_{i+1})}{(w - z_i)(tw - t^{-1}z_{i+1})} = \Delta_t(z_i, z_{i+1})\left(\frac{1}{(w - z_i)(w - z_{i+1})} + \frac{1}{(tw - t^{-1}z_i)(tw - t^{-1}z_{i+1})}\right). \quad (10)$$

This observation naturally leads to integral formulas using Bethe Ansatz functions, see e.g. [31] for similar formulas in the context of solutions to the q-deformed Knizhnik-Zamolodchikov equations. First we define

$$\phi_i(w) = \prod_{m=1}^{i} \frac{1}{w - z_m} \prod_{m=i+1}^{N} \frac{1}{tw - t^{-1}z_m}, \quad (11)$$

and introduce the shift operator S_i which acts as

$$S_i\phi_i = \phi_{i-1}, \qquad S_i\phi_j = \phi_j \quad (j \neq i). \quad (12)$$

Using (10), we have the following simple action,

$$T_i(u_k + 1)\Delta_t(z_1, \ldots, z_N)\phi_i(w) = \Delta_t(z_1, \ldots, z_N)\Big(S_i + S_i^{-1} + y_k\Big)\phi_i(w),$$

$$T_i(u_k + 1)\Delta_t(z_1, \ldots, z_N)\phi_j(w) = \frac{[2 + u_k]}{[1 + u_k]}\Delta_t(z_1, \ldots, z_N)\phi_j(w) \qquad (j \neq i),$$

where

$$y_k = -\frac{[u_k]}{[u_k+1]}.$$

The action of the Hecke generators on the "wave functions" ϕ_i leads to the following result, in which the deformed M polynomials are explicitly given as multiple integrals.

Theorem 1. *Let $a_k = \lambda_k + k$. We have*

$$\begin{aligned}
\mathrm{M}_{[\lambda_n,\ldots,\lambda_1]}(u_1,\ldots,u_n) &= \Delta_t(z_1,\ldots,z_N)\oint\frac{\mathrm{d}w_1}{2\pi\mathrm{i}}\cdots\oint\frac{\mathrm{d}w_n}{2\pi\mathrm{i}}\ \Delta(w_n,\ldots,w_1)\Delta_t(w_1,\ldots,w_n)\times \\
&\qquad \prod_{m=1}^{n}(1+y_mS_{a_m})\ \phi_{a_m}(w_m) \\
&= \Delta_t(z_1,\ldots,z_N)\oint\frac{\mathrm{d}w_1}{2\pi\mathrm{i}}\cdots\oint\frac{\mathrm{d}w_n}{2\pi\mathrm{i}}\ \Delta(w_n,\ldots,w_1)\Delta_t(w_1,\ldots,w_n)\times \\
&\qquad \prod_{m=1}^{n}\frac{1}{[u_m+1]}\left(\frac{t^{u_m+1}w_m - t^{-u_m-1}z_{a_m}}{tw_m - t^{-1}z_{a_m}}\right)\phi_{a_m}(w_m).
\end{aligned}$$

Proof. The proof is inductive and given in the appendix. □

3.2. Expansion of M into KL polynomials

Having the integrals of Theorem 1 at our disposal, we now investigate the expansion of M-basis elements into the KL basis.

Theorem 2. *Let $[\mu] = [n,\ldots,1]$ be the staircase partition. The deformed maximal Macdonald polynomial $M_{[\mu]}(u_1,\ldots,u_n)$ is equal to the sum*

$$M_{[\mu]}(u_1,\ldots,u_n) = \sum_{[\lambda]\subseteq[\mu]} c_\lambda\ \mathrm{KL}_{[\lambda]},$$

where the coefficients c_λ are monomials in $y_1,\ldots,y_n$ of degree at most 1 in each variable, and each KL element appears in the sum. These coefficients are recursively obtained by decomposing diagrams into ribbons, using equations (14) *and* (15) *below.*

Before proving this theorem, we first give an example.

Example 7 ($N = 2n+2 = 6$)**.** *Let*

$$\mathrm{M}_{[2,1]}(u_1,u_2) = T_2(u_1+1)T_4(u_2+1)T_3(u_2+2)\ \Delta_t(z_1,z_2,z_3)\Delta_t(z_4,z_5,z_6).$$

The integral representation for $\mathrm{M}_{[2,1]}(u_1,u_2)$ is

$$\begin{aligned}
\mathrm{M}_{[2,1]}(u_1,u_2) = \Delta_t(z_1,\ldots,z_6)\oint\frac{\mathrm{d}w_1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}w_2}{2\pi\mathrm{i}}\ &(w_2-w_1)(tw_1-t^{-1}w_2)\times \\
&(1+y_1S_2)\ (1+y_2S_4)\cdot\phi_2(w_1)\phi_4(w_2),
\end{aligned}$$

from which it immediately follows that

$$\begin{aligned}
\mathrm{M}_{[2,1]}(u_1,u_2) &= \mathrm{M}_{[2,1]}(0,0) + y_1\mathrm{M}_{[2]}(\cdot,0) + y_2\mathrm{M}_{[1,1]}(0,0) + y_1y_2\mathrm{M}_{[1]}(\cdot,0), \\
&= \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} + y_1\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} + y_2\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array} + y_1y_2\begin{array}{|c|}\hline 1 \\ \hline\end{array} \\
&= \mathrm{KL}_{[2,1]} + y_1\mathrm{KL}_{[2]} + y_2(\mathrm{KL}_{[1,1]} + \mathrm{KL}_{\emptyset}) + y_1y_2\mathrm{KL}_{[1]}.
\end{aligned}$$

Proof of Theorem 2. The fact that the coefficients are polynomials in $y_1, \ldots, y_n$ of degree at most 1 in each variable follows immediately from the first line in Theorem 1. We will see below that they are in fact monomials.

We will use a recursive argument to prove that each KL element appears in the expansion. First note that the expansion of $\mathrm{M}[n, n-1, \ldots, 2, 0]$ can be obtained from that of $\mathrm{M}[n-1, \ldots, 1, 0]$ by a shift of index $\mathrm{KL}_\lambda \to \mathrm{KL}_{\lambda+1^{n-1}}$, i.e. if

$$\mathrm{M}_{[n-1,\ldots,1,0]} = \sum_{[\lambda]\subseteq[\mu]} c_\lambda \mathrm{KL}_{[\lambda]}, \tag{13}$$

then

$$\mathrm{M}_{[n,\ldots,2,0]} = \sum_{[\lambda]\subseteq[\mu]} c_\lambda \mathrm{KL}_{[\lambda+1^{n-1}]}. \tag{14}$$

This follows immediately by comparing the first line of Theorem 1 with and without shifted indices $\lambda_k \to \lambda_k + 1$, $k = 1, \ldots, n$.

Secondly, the polynomial $\mathrm{M}_{[n,\ldots,1]}$ is the image of $\mathrm{M}_{[n,\ldots,2,0]}$ under the action of $T_2(u_1+1)$, i.e.

$$\mathrm{M}_{[n,\ldots,1]}(u_1, \ldots, u_n) = T_2(u_1+1)\mathrm{M}_{[n,\ldots,2,0]}(u_2, \ldots, u_n),$$

and we can use the known action of $T_2(u_1+1) = T_2(1) + y_1$ on the KL polynomials appearing in the expansion of $\mathrm{M}_{[n,\ldots,2,0]}$. In order to explain this in terms of partitions, let us first define the notion of a Dyck ribbon.

Given a partition $[\lambda]$ and a diagonal, a Dyck ribbon is a a strip μ of boxes on the border of $[\lambda]$ which starts and ends on the diagonal and never crosses it, and for which the shape of $[\lambda/\mu]$ is again a partition. The example in Figure 3.2 illustrates this notion pictorially, see also [32].

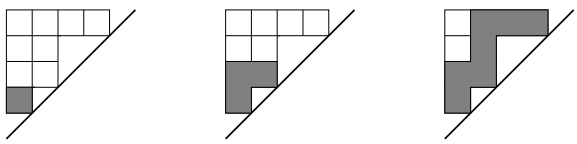


Figure 2: The partition [4,2,2,1] with diagonal as shown has three Dyck ribbons.

Let $[\lambda] = [\lambda_1, \ldots, \lambda_k]$ be a partition, $[\lambda']$ its dual, and let $[\rho]$ be the staircase $[\lambda'_1+1, \lambda'_1, \ldots, 1]$. Let R_ρ denote the set of Dyck ribbons of $[\lambda]$ with diagonal the diagonal of $[\rho]$. Then the action of $T_2(u_1+1) = T_2(1) + y_1$ on $\mathrm{KL}_{[\lambda]}$ is given by [22]

$$T_2(u_1+1)\ \mathrm{KL}_{[\lambda]} = \mathrm{KL}_{[\lambda,1]} + y_1\mathrm{KL}_{[\lambda]} + \sum_{\mu\in R_\rho} \mathrm{KL}_{[\lambda/\mu]}, \tag{15}$$

where the last sum only appears if $\lambda_k = 1$.

Example 8.

$$\begin{aligned} T_2(u_1+1)\mathrm{KL}_{[2,2]} &= \mathrm{KL}_{[2,2,1]} + y_1\mathrm{KL}_{[2,2]}, \\ T_2(u_1+1)\mathrm{KL}_{[2,1]} &= \mathrm{KL}_{[2,1,1]} + y_1\mathrm{KL}_{[2,1]} + \mathrm{KL}_{[2]} + \mathrm{KL}_\emptyset. \end{aligned}$$

Equations (14) and (15) give rise to a recursive procedure to determine the coefficients c_λ in the expansion (13). A hint of how the recursion works is obtained by considering the explicit the expansion of $\mathrm{M}_{[3,2,1]}$ using (15), given in Appendix C.

A recursion for c_λ.
A recursion for the coefficient c_λ may be deduced from (14) and (15), which is best explained in an example. We therefore show in Example 9 below how to recursively find the coefficient $c_{[2,2]}$ of $\mathrm{KL}_{[2,2]}$ in the expansion of $\mathrm{M}_{[6,5,4,3,2,1]}$. From this example it will be clear how to find the coefficient c_λ for an arbitrary partition λ in the expansion of any deformed maximal M polyomial.

Example 9. *Consider the Macdonald polynomial* $\mathrm{M}_{[6,5,4,3,2,1]}$*, i.e. corresponding to the maximal staircase of size* $n = 6$*. We wish to find the coefficient of* $\mathrm{KL}_{[2,2]}$ *in the expansion of this polynomial. To do so, we begin by drawing the partition* $[2, 2]$ *inside the staircase.*

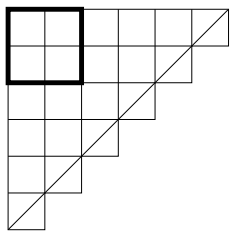
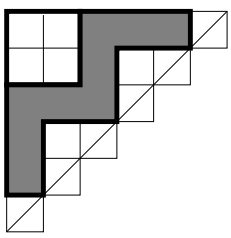


Figure 3: The maximal staircase of size $n = 6$, with inner shape $[2, 2]$

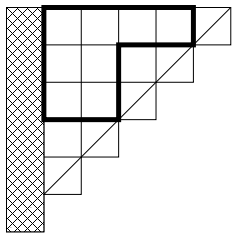
We now show the recursion step by step.

• First iteration

Step 1*: We draw in the diagonal bounding the staircase, and add the maximal Dyck ribbon between the diagonal and the inner shape (if possible). This is shown in Figure 4(a).*
Step 2*: The second step is to delete the first column. Our inner shape is now the remainder of* $[2, 2]$ *and the added Dyck ribbon, i.e. the new inner shape becomes* $[4, 2, 2]$*, see Figure 4(b).*



(a) The maximal Dyck ribbon is shaded



(b) The first column is removed, and new inner shape drawn

Figure 4: Steps 1 and 2 in the first iteration

• Second iteration

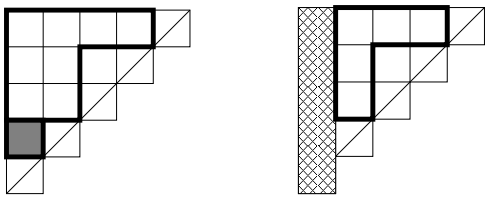
We now repeat steps 1 and 2, with the new inner shape and smaller staircase:
Step 1*: We draw in the diagonal bounding the staircase, and add the maximal Dyck ribbon in* $[6, 5, 4, 3, 2, 1]/[2, 2]$*. This is shown in Figure 5(a).*
Step 2*: We again remove the first column, see Figure 5(b), the new inner shape is* $[3, 1, 1]$*.*

• Third iteration

Step 1*: It is not possible to add a Dyck ribbon, see Figure 6(a). This has the consequence that now we pick up a factor of* y_3*.*
Step 2*: We again remove the first column, see Figure 6(b). The new inner shape is* $[2]$*.*

• Fourth iteration

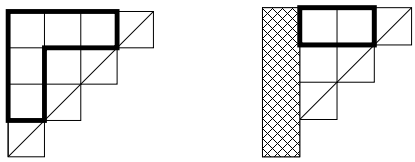
We repeat steps 1 and 2, with the new inner shape and smaller staircase:
Step 1*: We draw in the diagonal bounding the staircase, and add the maximal Dyck ribbon, see*



(a) The maximal Dyck ribbon is just the shaded single box

(b) The first column is again removed, and new inner shape drawn

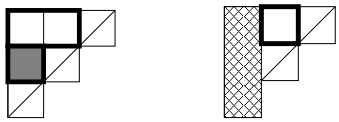
Figure 5: Steps 1 and 2 in the second iteration



(a) No Dyck ribbon can be added

(b) The first column is again removed

Figure 6: Steps 1 and 2 in the third iteration



(a) Maximal Dyck ribbon

(b) First column removed

Figure 7: Steps 1 and 2 in the fourth iteration

Figure 7(a).
Step 2*: We again remove the first column, see Figure 7(b). The new inner shape is* [1].

• Fifth and sixth iterations

Repeating steps we see that no further Dyck ribbon's can be added, see Figure 8. For these iterations therefore result in factors y_5 *and* y_6*:*

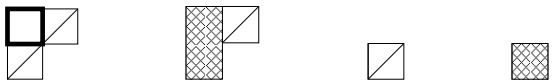


Figure 8: Steps 1 and 2 in the fifth and sixth iteration

Since we could not add Dyck ribbons at iterations 3, 5 and 6, we picked up corresponding factors y_3*,* y_5 *and* y_6*. The claim is that the coefficient of* $KL_{[22]}$ *in the expansion of* $M_{[654321]}$ *is just* $y_3y_5y_6$*.*

The entire preceding computation can be visualised in a single step, see Figure 9. In the first iteration, we fill the first Dyck ribbon with the numeral 1 *as a label of the iteration. We then*

consider the smaller staircase obtained by ignoring the first column. In the second iteration, the second Dyck ribbon is labelled with the numeral 2. This procedure continues for the six iterations. Where no Dyck ribbon can be added, a factor of y_i is added to the diagram.

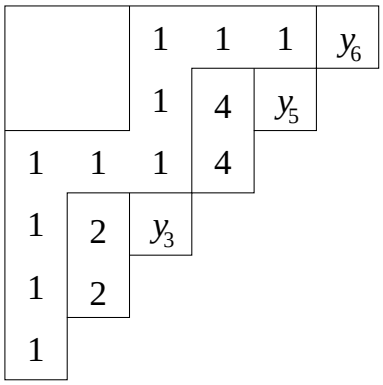


Figure 9: Summary of the coefficient computation

From Example 9 it is also clear that by repetition of steps 1 and 2, one may compute the coefficient of KL_λ for any λ in the expansion of M_ρ, with ρ a staircase. The corresponding coefficient c_λ is equal to the product of the factors y_i on the diagonal, showing that this coefficient is indeed a monomial. It is also clear that each c_λ is nonzero.

□

Corollary 1. *Equation* (7) *holds.*

Proof. First note that the case of equation (7) corresponds to $y_i = \tau^{-1} = -[2]^{-1}$ for $i = 1, \ldots, n$. Secondly, assigning signs as in Figure 1, the signs within each Dyck ribbon in the recursion above add up to zero. The sum over signs between inner shape and staircase is therefore the same as the number of boxes containing a factor y_i, and hence the total coefficient c_λ for each inner shape λ is equal to $c_\lambda = \tau^{-n_\lambda}$, where n_λ is the total sum of signed boxes. □

4. Constant term and Schubert polynomials

In this section we investigate the homogeneous limit $z_i \to 1$. The following change of variables is useful

$$z_i = \frac{1 - t^{-1} v_j}{1 - t v_j}, \qquad w_i = \frac{1 - t^{-1} x_j}{1 - t x_j}. \tag{16}$$

Defining

$$\tau = -(t + t^{-1}),$$

Proposition 1 (Homogenous limit)**.** *In the homogeneous limit $v_i \to 0$ we obtain*

$$\mathrm{M}_{[\lambda_n,\ldots,\lambda_1]}(u_1, \ldots, u_n) = C_{N,n} \oint \ldots \oint \prod_{m=1}^{n} \frac{\mathrm{d}x_k \; (1 + y_m x_m)}{2\pi \mathrm{i} \; x_m^{a_m}} \times$$
$$\prod_{1 \le i < j \le n} (x_j - x_i)(1 + x_i x_j + \tau x_j) \prod_{m=1}^{n} (1 - t x_m)^{N-2n} \; ,$$

where we recall that $a_k = \lambda_k + k$ and

$$C_{N,n} = (t - t^{-1})^{N(N-1)/2 + n(n+N)}, \qquad y_k = -\frac{[u_k]}{[u_k + 1]}.$$

In the case $N = 2n$ there is a further simplification.

A proof or Proposition 1 is given in the appendix.

The contour integrals in Proposition 1 pick out the residues around zero of each variable x_i. For simplicity we choose $N = 2n$ in which case the integral can be rewritten as the following constant term expression

$$\widetilde{N}_{a_1,\ldots,a_n}(\tau, y_1, \ldots, y_n) = \mathrm{CT}\left(\prod_{i=1}^{n}(1+y_i x_i)x_i^{-a_i+1}\prod_{i<j}(x_j - x_i)(1 + x_i x_j + \tau x_j)\right), \tag{17}$$

where CT = constant term with respect to each variable x_i. In the case $a_i = 2i-1$ and $y_1 = \ldots = y_n = y$, expression (17) is equal to a constant term expression given by Di Francesco and Zinn-Justin in [14]. In the same paper, a formula (Formula 2.7) is given for the number of TSSCPPs according to the heights of the vertical steps. This formula reads

$$N(t_0, \ldots, t_{n-1}) = \mathrm{CT}\left(\prod_{i<j}\frac{(x_j - x_i)(1 + t_i x_j)}{1 - x_i x_j}\prod_i \frac{1 + t_0 x_i}{1 - x_i^2}\prod_{i=1}^{n} x_i^{-2i+2}\right) \tag{18}$$

When $t_1 = \ldots t_n = \tau$, the two constant terms (17) and (18) are equal:

Theorem 3 (Zeilberger-Di Francesco-Zinn Justin)**.**

$$N(t, \tau, \ldots, \tau) = \widetilde{N}_{2,4,\ldots,2n}(\tau, t, \ldots, t).$$

Proof. This can be shown by symmetrising the argument of the constant term (17) with respect to the variables x_i, [14, 33]. □

We have not been able to establish a relation between N and $\widetilde{N}$ for general values of y_i and t_i. Fonseca and Zinn-Justin[27], and Fonseca and Nadeau [28] consider the number of punctured totally symmetric self complementary plane partition with a puncture of size r. Their generation function is equal to

$$N_{n,r}(\tau) = \mathrm{CT}\left(\prod_{i<j}\frac{(x_j - x_i)(1 + \tau x_j)}{1 - x_i x_j}\prod_i \frac{(1 + x_i)(1 + \tau x_i)^r}{1 - x_i^2}\prod_{i=1}^{n} x_i^{-2i+2}\right)$$

Theorem 4. *Let $[\mu_r]$ be the staircase partition $[n-r-1, n-r-2, 1, 0, \ldots, 0]$, and let μ_r be the corresponding Yamanouchi word. Then we have*

$$N_{n-r,r}(\tau) = M_{\mu_r}(u_i = 1; z_i = 1) = \widetilde{N}_{123\cdots,r-1,r+1,r+3,\ldots 2n-r-1}(\tau, 1, \ldots, 1).$$

Example 10. *For $n = 5$ and $r = 2$ we have $[\mu_2] = [2, 1]$, $\mu_2 = 4324130210$ and*

$$N_{3,2}(\tau) = M_{4324130210} = \widetilde{N}_{12357}(\tau, 1, 1, 1, 1, 1) = 1 + 7\tau + 12\tau^2 + 14\tau^3.$$

4.1. Schubert polynomials

We begin this section by recalling several facts about Schubert polynomials, (see e.g. [34]).

Let n be an integer, $\mathbf{x} = \{x_1, \ldots, x_n\}$, $\mathbf{y} = \{y_1, y_2, \ldots, y_\infty\}$. The *Schubert polynomials*, (sometimes called *double Schubert polynomials*), $Y_v(\mathbf{x}, \mathbf{y})$, $v \in \mathbb{N}^n$, are a linear basis of the space of polynomials in $\mathbf{x}$ with coefficients in $\mathbf{y}$. They are recursively defined starting from the case where v is *dominant*, i.e. $v = \lambda = [\lambda_1, \ldots, \lambda_n]$, $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n \geq 0$ (we also say that λ is a partition). In that case

$$Y_\lambda(\mathbf{x}, \mathbf{y}) := \prod_{i=1}^{n}\prod_{j=1}^{\lambda_i}(x_i - y_j)\,.$$

The general polynomials are then defined by

$$\partial_i Y_v(\mathbf{x}, \mathbf{y}) = Y_{[\ldots,v_{i-1},v_{i+1},v_i-1,v_{i+2},\ldots]}(\mathbf{x}, \mathbf{y})\ ,\ v_i > v_{i+1}\,, \tag{19}$$

(in the case where $v_i \leq v_{i+1}$, then $\partial_i Y_v(\mathbf{x},\mathbf{y}) = 0$ since $\partial_i\partial_i = 0$). The usual indexing of Schubert polynomials is by permutations in S_n. [1] In this case, the induction takes the more familiar form (see e.g. [34, 35]),

$$\begin{aligned} \partial_w Y_{w'} &= Y_{w'w^{-1}}, \quad \ell(w'w^{-1}) = \ell(w') - \ell(w), \\ &= 0, \qquad\qquad \text{otherwise.} \end{aligned} \tag{20}$$

Since the image of ∂_i is the space of polynomials which are symmetric in x_i, x_{i+1}, the Schubert polynomial $Y_v(\mathbf{x},\mathbf{y})$ has this symmetry whenever $v_i \leq v_{i+1}$. In particular, if v is such that $v_1 \leq \cdots \leq v_n$, then $Y_v(\mathbf{x},\mathbf{y})$ is symmetric in $x_1,\ldots,x_n$ and is called a *Graßmannian Schubert polynomial*. It specializes to the Schur function in $\mathbf{x}$ indexed by the partition $[v_n,\ldots,v_1]$ for $\mathbf{y} = \mathbf{0} = \{0,0,\ldots\}$.

Example 11. *The dominant Schubert polynomial can be visualised on the Young diagram corresponding to the index of the polynomial. Each box is filled with a factor of the form $(x_i - y_j)$, placed in the ith row and jth column. The Schubert polynomial is equal to the product of these factors. For $v = [4,2,1]$ the diagram is*

$x_1 - y_1$	$x_1 - y_2$	$x_1 - y_3$	$x_1 - y_4$
$x_2 - y_1$	$x_2 - y_2$		
$x_3 - y_1$			

so that we have

$$Y_{[4,2,1]} = (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_1-y_4)(x_2-y_1)(x_2-y_2)(x_3-y_1).$$

Example 12. *Consider the alphabets $\mathbf{x} = \{x_1,x_2,x_3\}$, $\mathbf{y} = \{y_1,y_2,y_3\}$. We find the Schubert polynomial $Y_{[0,1,0]}(\mathbf{x},\mathbf{y})$ from the dominant Schubert polynomial $Y_{[2,1,0]}(\mathbf{x},\mathbf{y})$:*

$$\begin{aligned} Y_{[0,1,0]}(\mathbf{x},\mathbf{y}) &= \partial_1\partial_2 Y_{[2,1,0]}(\mathbf{x},\mathbf{y}) \\ &= \partial_1\partial_2 \prod_{i+j\leq 3}(x_i - y_j) \\ &= \partial_1\partial_2 (x_1-y_1)(x_1-y_2)(x_2-y_1) \\ &= \partial_1 (x_1-y_1)(x_1-y_2) \\ &= x_1 + x_2 - y_1 - y_2. \end{aligned}$$

Following these preliminaries on Schubert polynomials, we now state

Theorem 5. *Let n be an integer, $\rho = [n-1,\ldots,1]$. For any $\lambda \subseteq \rho$, let $\mu = \lambda'$ be the conjugate partition, and let $(\rho/\lambda) = [0, n-1-\mu_1, 0, n-2-\mu_2, 0, n-3-\mu_3, 0, \ldots]$.*

[1] Recall that the index, or *code*, v, of a Schubert polynomial is related to a permutation σ by the sequence of numbers $v_i := \#(j : j > i, \sigma_j < \sigma_i)$. We can work from the leftmost integer, which gives the number of σ_j smaller than σ_1, hence determining the value of σ_1. We can then similarly obtain all σ_i of the permutation, working from left to right. For example, the code of the permutation (3142) is [2010]

Then the constant term (18) *is equal to*

$$N(t_0,\dots,t_{n-1}) = \sum_{\lambda\subseteq\rho,\ \lambda\ even} Y_{(\rho/\lambda)}(\bar{\mathbf{y}},0), \tag{21}$$

and the alphabet $\bar{\mathbf{y}}$ *being specialized into* $\{t_0,t_1,0,t_2,0,t_3,0,\dots\}$.

Proof of Theorem 5. Let us analyse the expression (18). One first extracts the kernel $\prod_{i<j}(1-x_i/x_j)$, then the symmetric function $\prod_{i\le j}(1-x_ix_j)^{-1}$. What remains can be identified with a Schubert polynomial. This allows us to use the following fundamental scalar product on polynomials : Given two polynomials f, g, one defines

$$\langle f,g\rangle = \mathrm{CT}\left(f(x_1,\dots,x_n)g(x_n^{-1},\dots,x_1^{-1})\prod_{1\le i<j\le n}(1-\frac{x_i}{x_j})\right). \tag{22}$$

It is proved in [36] that this scalar product coincides with the usual scalar product when restricted to symmetric functions. The function $N(t_0,\dots,t_{n-1})$ is now written

$$\begin{aligned} N(t_0,\dots,t_{n-1}) &= \mathrm{CT}\left(\prod_{i<j}(1-\frac{x_i}{x_j})\prod_{i\le j}(1-x_ix_j)^{-1}\prod_{i<j}(1+t_ix_j)\prod_{i=1}^{n}(1+t_0x_i)\prod_{i=1}^{n}x_i^{-i+1}\right)\\ &= \left\langle \prod_{i\le j}(1-x_ix_j)^{-1},\prod_{i<j}(1+\frac{t_i}{x_{n+1-j}})\prod_{i=1}^{n}(1+\frac{t_0}{x_i})x^{\rho_i}\right\rangle\\ &= \left\langle \prod_{i\le j}(1-x_ix_j)^{-1},\prod_{i+j\le n}(x_i+t_j)\prod_{i=1}^{n-1}(1+\frac{t_0}{x_i})\right\rangle \end{aligned} \tag{23}$$

as the term t_0/x_n does not contribute to the constant term.

Recall that the operator $\pi_\omega:\ f\mapsto\partial_\omega(x^\rho f)$ is self adjoint and sends a dominant monomial x^λ onto the Schur function of the same index. Since the LHS of the scalar product is symmetric, one can symmetrize the RHS under π_ω, transforming it into

$$\partial_w\left(\prod_{i+j\le n}(x_i+t_j)\prod_{i=1}^{n-1}(1+\frac{t_0}{x_i})x_i^{\rho_i}\right). \tag{24}$$

However

$$\prod_{i+j\le n}(x_i+t_j)\prod_{i}^{n-1}(1+\frac{t_0}{x_i})x_i^{\rho_i}$$

is equal to the dominant Schubert polynomial of index $[2n-2,2n-4,\dots,2,0]$ for the alphabets $\mathbf{x}=\{x_1,\dots,x_n\}$, $\mathbf{y}=\{-t_0,-t_1,0,-t_2,0,-t_3,\dots\}$. For example, for $n=4$, this Schubert polynomial is

x_1+t_0	x_1+t_1	x_1+0	x_1+t_2	x_1+0	x_1+t_3
x_2+t_0	x_2+t_1	x_2+0	x_2+t_2		
x_3+t_0	x_3+t_1				

We can therefore write

$$N(t_0,\dots,t_{n-1}) \quad = \quad \left\langle \prod_{i\le j}(1-x_ix_j)^{-1}, \partial_\omega Y_{[2n-2,2n-4,\dots,2,0]}(\mathbf{x},\mathbf{y}) \right\rangle. \tag{25}$$

Using $\partial_\omega = \partial_{n-1}\circ\partial_{n-2}\partial_{n-1}\circ\cdots\circ\partial_2\partial_3\dots\partial_{n-2}\partial_{n-1}$ and (19), we can simplify the RHS of the scalar product with

$$\partial_\omega Y_{[2n-2,2n-4,\dots,2,0]}(\mathbf{x},\mathbf{y}) = Y_{[0,1,\dots,n-1]}(\mathbf{x},\mathbf{y}).$$

There is a Cauchy formula for Schubert polynomials [34, (Theorem 10.2.6)], which decomposes a Schubert polynomial into an alternating sum of products of Schubert polynomials in each of the alphabets $\mathbf{x}$, $\mathbf{y}$ separately. In our case, $Y_{[0,1,\dots,n-1]}(\mathbf{x},\mathbf{y})$ is symmetric in $x_1,\dots,x_n$ and decomposes into a sum of Schur functions in $\mathbf{x}$:

$$Y_{[0,1,2,\dots,n-1]}(\mathbf{x},\mathbf{y}) = \sum_{\lambda\subseteq\rho} S_\lambda(\mathbf{x})Y_{(\rho/\lambda)}(\bar{\mathbf{y}},0), \tag{26}$$

the definition of the index (ρ/λ) and the alphabet $\bar{\mathbf{y}}$ being given in the statement of the theorem.

The LHS of the scalar product is equal to the sum of all Schur functions indexed by an even partition (i.e. with even parts). Therefore, the scalar product is equal to

$$N(t_0,\dots,t_{n-1}) = \sum_{\lambda\subseteq\rho,\ \lambda\ even} Y_{(\rho/\lambda)}(\bar{\mathbf{y}},0).$$

□

Example 13.

For $n=3$, we have $\rho=[2,1]$. The two possible even sub-partitions are $\lambda=\emptyset$ and $\lambda=[2]$. The two conjugate partitions are $\mu=\lambda'=\emptyset$, and $\mu=\lambda'=[1,1]$ respectively. Considering the first of these we have

$$(\rho/\lambda) = (21/0) = [0, n-1-\mu_1, 0, n-2-\mu_2, 0] = [0,2,0,1,0],$$

and for the second sub-partition we have

$$(\rho/\lambda) = (21/2) = [0, n-1-\mu_1, 0, n-2-\mu_2, 0] = [0,1,0,0,0].$$

Thus the two polynomials contributing to the constant term $N(t_0,t_1,t_2)$ are

$$Y_{[0,2,0,1,0]}(\bar{\mathbf{y}},0) \text{ and } Y_{[0,1,0,0,0]}(\bar{\mathbf{y}},0)$$

4.2. Determinant expression

The Schubert polynomials in Theorem 5 have a determinantal expression that we are now going to introduce. Let $h_k(r)$ denote the complete symmetric function of degree k in the alphabet $\bar{\mathbf{y}} = \{y_1,\dots,y_{2r-2}\} = \{t_0,t_1,0,t_2,0,t_3,\dots,t_{r-1}\}$. (Recall that the complete symmetric function of degree k is defined to be the sum of all monomials of total degree k – see e.g. [37]). We also use the notation $\mu=\lambda'$.

The indices of the Schubert polynomials in $\bar{\mathbf{y}}$ on the RHS of (21), are all codes which can be obtained from an increasing partition under the Schubert recursion (19). Therefore, these polynomials are all images of Schur functions in $\bar{\mathbf{y}}$ under divided differences in $\bar{\mathbf{y}}$.

Given a Schur function, one writes it as a determinant over complete symmetric functions, but with a flag of alphabets, see e.g. [34, Lemma 1.4.5] or [35, Corollary 2.6.10]. The images of such a determinant under any product of divided difference is easy to compute. At each step any divided difference acts on a single row only (or on no row). From (19) it follows that this action consists in decreasing the indices of the complete symmetric functions in this row and transforming their argument r into $r+1$. We thus obtain

Proposition 2. *Let $\mu = \lambda'$ be the partition conjugate to λ. Then*

$$Y_{(\rho/\lambda)}(\bar{\mathbf{y}}, 0) = S_{\rho/\lambda}(y_{\phi_1}, y_{\phi_2}, \ldots) = \det\left(h_{\rho_i - \mu_j - i + j}(\phi_i)\right)_{i,j=1\ldots n-1} \tag{27}$$

where the flag is equal to $\phi_i = 2i$. Equivalently, suppressing the 0's in $\bar{\mathbf{y}}$ and using the alphabet $\tilde{\mathbf{y}} = (t_0, t_1, t_2, t_3, \ldots)$, one has

$$Y_{(\rho/\lambda)}(\tilde{\mathbf{y}}, 0) = S_{\rho/\lambda}(\tilde{y}_{\tilde{\phi}_1}, \tilde{y}_{\tilde{\phi}_2}, \ldots) = \det\left(\tilde{h}_{\rho_i - \mu_j - i + j}(\tilde{\phi}_i)\right)_{i,j=1\ldots n-1} \tag{28}$$

where the flag is equal to $\tilde{\phi}_i = i + 1$.

We illustrate this is in the following examples.

Example 14.

$$Y_{[4,6,7]}(\bar{\mathbf{y}}, 0) = \begin{vmatrix} h_4(1) & h_7(1) & h_9(1) \\ h_3(2) & h_6(2) & h_8(2) \\ h_2(3) & h_5(3) & h_7(3) \end{vmatrix}.$$

We transform this into $Y_{[0,3,0,4,0,4]}$ using the sequence of divided difference operators $\partial_1\partial_3\partial_2\partial_5\partial_4\partial_3$. The first step gives

$$Y_{[4,6,7]} \xrightarrow{\partial_3} Y_{[4,6,0,6]} = \begin{vmatrix} h_4(1) & h_7(1) & h_9(1) \\ h_3(2) & h_6(2) & h_8(2) \\ h_1(4) & h_4(4) & h_6(4) \end{vmatrix},$$

and the remaining transformations lead to

$$Y_{[4,6,0,6]} \xrightarrow{\partial_4} Y_{[4,6,0,0,5]} \xrightarrow{\partial_5} Y_{[4,6,0,0,0,4]} \xrightarrow{\partial_2} Y_{[4,0,5,0,0,4]} \xrightarrow{\partial_3} Y_{[4,0,0,4,0,4]} \xrightarrow{\partial_1} Y_{[0,3,0,4,0,4]} = \begin{vmatrix} h_3(2) & h_6(2) & h_8(2) \\ h_1(4) & h_4(4) & h_6(4) \\ 0 & h_2(6) & h_4(6) \end{vmatrix}.$$

Recall that we have taken the alphabet $\bar{\mathbf{y}} = (t_0, t_1, 0, t_2, \ldots)$, and the flag consists of exactly the positions of the non-zero components of the index. Let $\tilde{h}_i(r)$ be the complete symmetric function in the alphabet $\tilde{\mathbf{y}}$. Then

$$Y_{[0,3,0,4,0,4]} = \begin{vmatrix} \tilde{h}_3(2) & \tilde{h}_6(2) & \tilde{h}_8(2) \\ \tilde{h}_1(3) & \tilde{h}_4(3) & \tilde{h}_6(3) \\ 0 & \tilde{h}_2(4) & \tilde{h}_4(4) \end{vmatrix}.$$

Example 15. *Using* (28)*, we write*

$$\begin{aligned} Y_{[0,2,0,1,0]}(\bar{\mathbf{y}}, 0) &= \begin{vmatrix} h_2(2) & h_3(2) \\ h_0(3) & h_1(3) \end{vmatrix} \\ &= \begin{vmatrix} (t_0^2 + t_0t_1 + t_1^2) & (t_0^3 + t_0^2t_1 + t_0t_1^2 + t_1^3) \\ 1 & (t_1 + t_1 + t_2) \end{vmatrix} \\ &= t_0^2t_1 + t_0^2t_2 + t_0t_1^2 + t_0t_1t_2 + t_1^2t_2. \end{aligned}$$

Similarly

$$\begin{aligned} Y_{[0,1,0,0,0]}(\bar{\mathbf{y}}, 0) &= \begin{vmatrix} h_1(2) & h_2(2) \\ h_{-1}(3) & h_0(3) \end{vmatrix} \\ &= \begin{vmatrix} (t_0 + t_1) & (t_0^2 + t_0t_1 + t_1^2) \\ 0 & 1 \end{vmatrix} \\ &= t_0 + t_1. \end{aligned}$$

and thus

$$N(t_0, t_1, t_2) = Y_{[0,2,0,1,0]}(\bar{\mathbf{y}}, 0) + Y_{[0,1,0,0,0]}(\bar{\mathbf{y}}, 0) = t_0 + t_1 + t_0^2t_1 + t_0^2t_2 + t_0t_1^2 + t_0t_1t_2 + t_1^2t_2. \tag{29}$$

In terms of permutations, the reduced decompositions corresponding to (ρ/λ) may be simply obtained using the following procedure: start with a canonical filling of the staircase

n	$n{+}1$	$n{+}2$	$\cdots$	$\cdots$	$2n{-}2$
$n{-}1$	n	$n{+}1$	$\cdots$	$2n{-}4$	
3	4				
2					

and erase, starting from the north-west corner, the diagram of a partition with rows of even lengths. The numbers of remaining entries in each column give the index of the Schubert polynomial. The numbers in the boxes, read by column, give the reduced decomposition.

Example 16. *Consider the Schubert polynomial index* $[0,1,0,0,0]$*. The permutation corresponding to this code is* $[1,3,2,4]$*. A reduced decomposition of this permutation is* $(2,3)$*, which we will write as* 2*. Similarly the index* $[0,2,0,1,0]$ *is the code of the permutation* $(1,4,2,5,3)$*. A reduced decomposition of this permutation is* $(3,4),(2,3),(4,5)$*, which we denote as* 324*.*
Consider now $n=3$*. We have the following staircase:*

3	4
2	

For the even partition $\lambda=0$*, no boxes are erased. Hence we still have* 2 *boxes in column* 1 *and* 1 *box in column* 2*. This corresponds to an index of* $[0,2,0,1,0]$*. Reading the entries from left to right by column, we obtain the integers* $3,2,4$*, giving the reduced decomposition given above.*
For the even partition $\lambda=2$*, we erase the two boxes from the first row. This leaves only a single box in the first column, hence corresponding to an index of* $[0,1,0,0,0]$*. The remaining box contains the integer* 2*, also agreeing with the reduced decomposition given above.*

4.3. Combinatorial interpretation

Having interpreted the constant term in terms of Schubert polynomials, one gets for free a combinatorial interpretation in terms of tableaux. Indeed, $Y_{[0,n-1,0,\dots,3,0,2,0,1]}(\bar{\mathbf{y}})$ is equal to a sum of semi-standard tableaux of staircase shape, that we represent in the French way in the Cartesian plane, satisfying a flag condition. Taking the alphabet $\tilde{\mathbf{y}}=(t_0,t_1,t_2,t_3,\dots)$, the Schubert polynomial becomes $Y_{[0,n-1,n-2,\dots,2,1]}(\tilde{\mathbf{y}})$, which can be interpreted as the sum of tableaux of staircase shape such that the bottom row belongs to $\{1,2\}^*$, i.e only t_0 and t_1 can be used as fillings for this row, the next one to $\{1,2,3\}^*,\dots,$ the top one to $\{1,\dots,n\}^*$. We say that such a tableau satisfies the flag condition $2,3,\dots,n$. For $n=4$, this is

			flag
			$1,2,3,4$
			$1,2,3$
			$1,2$

This interpretation remains valid for skew shapes. As usual in the theory of tableaux, this result is obtained by introducing an extra alphabet of small letters, which fill the inner shape. The

valuation of these tableaux for a fixed inner shape with even columns are exactly the determinants written above.
Therefore, (21) becomes

Theorem 6. *The constant term $N(t_0, \ldots, t_{n-1})$ is the commutative image of the sum of tableaux of outer shape $\rho = [n-1, \ldots, 1]$, inner shape with columns of even lengths, satisfying the flag condition $2, 3, \ldots, n$.*

In fact, one can reprove directly that the generating function $N(t_0, \ldots, t_{n-1})$ is given by a sum of tableaux.

Example 17. *For $n = 3$, the sum*

$$Y_{(21/0)}(\bar{\mathbf{y}}, 0) + Y_{(21/2)}(\bar{\mathbf{y}}, 0) = Y_{[0,2,0,1,0]}(\bar{\mathbf{y}}, 0) + Y_{[0,1,0,0,0]}(\bar{\mathbf{y}}, 0)$$

is given by the tableaux

$$\begin{array}{|c|c|}\hline t_2 & \\ \hline t_0 & t_0 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline t_2 & \\ \hline t_0 & t_1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline t_2 & \\ \hline t_1 & t_1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline t_1 & \\ \hline t_0 & t_0 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline t_1 & \\ \hline t_0 & t_1 \\ \hline \end{array} \quad \begin{array}{|c|}\hline t_0 \\ \hline \end{array} \quad \begin{array}{|c|}\hline t_1 \\ \hline \end{array}$$

and this agrees with

$$N(t_0, t_1, t_2) = t_0^2 t_2 + t_0 t_1 t_2 + t_1^2 t_2 + t_0^2 t_1 + t_0 t_1^2 + t_0 + t_1.$$

We now introduce another object which is in bijection with TSSCPP. We take the staircase shape now in the north-west corner (here for $n = 5$):

$$\begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & & \\ \square & & & \end{array} \qquad \text{with border:} \qquad \begin{array}{ccccc} \square & \square & \square & \square & (0) \\ \square & \square & \square & (1) & \\ \square & \square & (2) & & \\ \square & (3) & & & \end{array}$$

The TSSCPP are in bijection with the fillings of the completed staircase such that rows are weakly decreasing from left to right, and decrease by 1 at most (taking into account the border). Each filling corresponds to a configuration of non-intersecting lattice paths, beginning at $(i, -i)$, obtained by reading each row from right to left, and an increase in the integer corresponds to a vertical step.

Example 18. *For the filling*

$$\begin{array}{ccccc} 1 & 1 & 1 & 0 & (0) \\ 2 & 2 & 1 & (1) & \\ 3 & 3 & (2) & & \\ 3 & (3) & & & \end{array}$$

we obtain the non-intersecting lattic paths (NILP) shown in Figure 10.

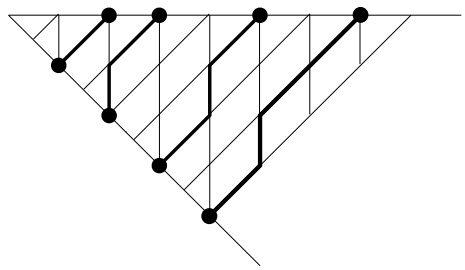


Figure 10: Non-intersecting lattice path configuration. Reading each row from right to left, an increase in value corresponds to a vertical step

Now add a left column (column 0) as follows. If the number of rows is even (*resp. odd*), then it consists of the even (*resp. odd*) numbers immediately bigger than or equal to the numbers in the first column. For example,

$$\begin{array}{cccccc} 1 & 1 & 1 & 0 & (0) \\ 2 & 2 & 1 & (1) & \\ 3 & 3 & (2) & & \\ 3 & (3) & & & \end{array} \quad \text{completes into} \quad \begin{array}{cccccc} 2 & 1 & 1 & 1 & 0 & (0) \\ 2 & 2 & 2 & 1 & (1) & \\ 4 & 3 & 3 & (2) & & \\ 4 & 3 & (3) & & & \end{array}$$

This corresponds to the completion defined by Di Francesco and Zinn-Justin in [14], and discussed in Section 2.3 (see Example 5). Again the rows describe the successive paths (read from right to left) composing the TSSCPP, an increase corresponding to a vertical step.

Example 19. *For the filling, with completion described above, we have the path configuration shown in Figure 11.*

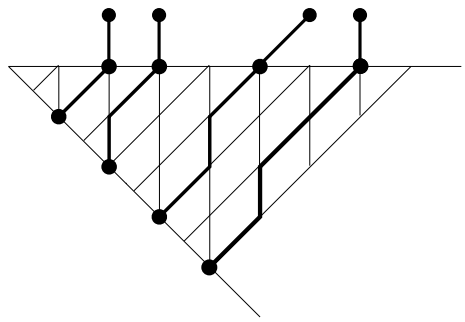


Figure 11: Augmented non-intersecting lattice path configuration. Again reading each row from right to left, and an increase in value corresponds to a vertical step

But such an object can also be read as a usual skew Young tableau (in the Cartesian plane, with strictly decreasing columns, and weakly increasing rows) of outer shape the staircase, with inner shape the diagram with lengths of columns the complement of column 0 to $[(n-1)^{n-1}]$. In the example above the inner shape would be $[4444]-[2244]=[2200]$. One records the positions of the decreases in row $1,2,\ldots,$ (reading left to right), and this gives columns $1,2,\ldots$ of the tableau (reading bottom to top). Such a tableau satisfies the flag condition $2,3,\ldots,$ its commutative evaluation is exactly the weight of the corresponding TSSCPP (where t_0 is the weight of each augmented vertical step, t_1 the weight of each vertical step in the row immediately below this,

etc.). The example becomes

4			
1	3		
·	·	3	
·	·	1	1

and thus has a weight $t_0^3 t_2^2 t_3$.

In summary, each weighted NILP corresponds to a filling, as described above. Each filling corresponds to a skew tableau, satisfying a flag condition, describing a monomial of the Schubert polynomial $Y_{(\rho/\lambda)}(\bar{\mathbf{y}}, 0)$ for some even partition λ.

Acknowledgement

AL and JdG would like to thank the hospitality of the Mathematisches Forschungsinstitut Oberwolfach, where part of this work was done, and we have greatly benefited from discussions with Pavel Pyatov and Keiichi Shigechi. We thank the Australian Research Council (ARC) for financial support.

Appendix A. Proof of Theorem 1

Appendix A.1. One row

First we recall

$$\phi_i(w) = \prod_{m=1}^{i} \frac{1}{w - z_m} \prod_{m=i+1}^{N} \frac{1}{tw - t^{-1} z_m}, \tag{A.1}$$

and we have

$$T_i(u_1)\Delta_t(z_1, \ldots, z_N)\phi_i(w) = \Delta_t(z_1, \ldots, z_N)\Big(\phi_{i-1}(w) + \phi_{i+1}(w) + \frac{[1-u_1]}{[u_1]}\phi_i(w)\Big), \tag{A.2}$$

$$T_i(u_1)\Delta_t(z_1, \ldots, z_N)\phi_j(w) = \frac{[1+u_1]}{[u_1]}\Delta_t(z_1, \ldots, z_N)\phi_j(w) \qquad (j \neq i). \tag{A.3}$$

We now look for a solution ψ_i of (A.2) and (A.3) satisfying the boundary conditions $\psi_0 = \psi_{N+1} = 0$. These boundary conditions can be fixed using Cauchy's theorem. Let C_0 denote the contour lying around all poles at z_i $(i = 1, \ldots, N)$, and let C_1 be the contour lying around the poles at $t^{-2} z_i$. As $\phi_i(w)$ has no pole at infinity, the contour C_0 may be deformed to C_1, and we have

$$\oint_{C_0} \frac{\mathrm{d}w}{2\pi \mathrm{i}}\, \phi_i(w) = -\oint_{C_1} \frac{\mathrm{d}w}{2\pi \mathrm{i}}\, \phi_i(w). \tag{A.4}$$

By Cauchy's theorem, the first integral obviously gives zero for $i = 0$, and second is clearly zero for $i = N$.

Let us now define

$$\psi_i = \Delta_t(z_1, \ldots, z_N) \oint_{C_0} \frac{\mathrm{d}w}{2\pi \mathrm{i}}\, \phi_i(w), \tag{A.5}$$

and note that

$$\psi_2(u_1) := T_1(u_1 + 1)\psi_1 = \psi_2 - \frac{[u_1]}{[u_1 + 1]}\psi_1.$$

Let us further introduce the shift operator S_i which acts as

$$S_i\phi_i = \phi_{i-1}, \qquad S_i\phi_j = \phi_j \quad (j \neq i). \tag{A.6}$$

We then have that

$$\begin{aligned}\psi_2(u_1) &= \Delta_t(z_1,\dots,z_N)\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}}\ \left(1-\frac{[u_1]}{[u_1+1]}S_2\right)\ \phi_2(w)\\ &= \Delta_t(z_1,\dots,z_N)\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}}\ \left(\frac{1}{[u_1+1]}\frac{t^{u_1+1}w-t^{-u_1-1}z_2}{tw-t^{-1}z_2}\right)\phi_2(w).\end{aligned}$$

More generally we define

$$\psi_{k+1}(u_1) = T_k(u_1+1)\cdots T_1(u_1+k)\psi_1, \tag{A.7}$$

which we denote by

$$\psi_{k+1}(u_1) = \begin{array}{|c|c|c|c|c|c|}\hline u_1+k & & & & u_1+2 & u_1+1\\ \hline\end{array}\ .$$

Lemma 1. *For $a \geq 2$,*

$$\begin{aligned}\psi_a(u_1) &= \Delta_t(z_1,\dots,z_N)\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}}\ \left(1-\frac{[u_1]}{[u_1+1]}S_a\right)\ \phi_a(w)\\ &= \Delta_t(z_1,\dots,z_N)\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}}\ \left(\frac{1}{[u_1+1]}\frac{t^{u_1+1}w-t^{-u_1-1}z_a}{tw-t^{-1}z_a}\right)\phi_a(w).\end{aligned}$$

Proof. The proof is by induction. From (A.7) it follows that

$$\psi_{a+1}(u_1) = T_a(u_1+1)\psi_a(u_1+1).$$

Using the properties (A.2) and (A.3) it is then easy to see that

$$\begin{aligned}\psi_{a+1}(u_1) &= \oint \frac{\mathrm{d}w}{2\pi\mathrm{i}}\ T_a(u_1+1)\Delta_t(z_1,\dots,z_N)\left(\phi_a(w)-\frac{[u_1+1]}{[u_1+2]}\phi_{a-1}(w)\right)\\ &= \Delta_t(z_1,\dots,z_N)\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}}\ \left(\phi_{a-1}(w)+\phi_{a+1}(w)-\frac{[u_1]}{[u_1+1]}\phi_a(w)-\phi_{a-1}(w)\right)\\ &= \Delta_t(z_1,\dots,z_N)\oint \frac{\mathrm{d}w}{2\pi\mathrm{i}}\ \left(1-\frac{[u_1]}{[u_1+1]}S_{a+1}\right)\phi_{a+1}(w).\end{aligned}$$

□

Appendix A.2. Two rows

Now we consider the function

$$\phi_{k_1k_2}(w_1,w_2) = \Delta_t(z_1,\dots,z_N)\ A(w_1,w_2)\ \phi_{k_1}(w_1)\phi_{k_2}(w_2). \tag{A.8}$$

We then find that

$$T_2(u_2)\phi_{12}(w_1,w_2) = \phi_{11}(w_1,w_2)+\phi_{13}(w_1,w_2)+\frac{[1-u_2]}{[u_2]}\phi_{12}(w_1,w_2).$$

The "unwanted term" $\phi_{11}(w_1,w_2)$ can be made to vanish if we take $A(w_1,w_2)\propto w_1-w_2$ and integrate both w_1 and w_2 around the point z_1. Let therefore

$$\psi_{12} = \Delta_t(z_1,\dots,z_N)\oint \frac{\mathrm{d}w_1}{2\pi\mathrm{i}}\ \oint \frac{\mathrm{d}w_1}{2\pi\mathrm{i}}\ \phi_{12}(w_1,w_2,), \tag{A.9}$$

where the integration contour encircles the poles at $w = z_i$ but not those at $w = t^{-2}z_i$. In order to satisfy the initial condition

$$\psi_{12} = \Delta_t(z_1, z_2)\Delta_t(z_3, \ldots, z_N), \tag{A.10}$$

we find that

$$A(w_1, w_2) = (w_2 - w_1)(tw_1 - t^{-1}w_2). \tag{A.11}$$

Finally we define

$$\psi_{k_1+1,k_2+2}(u_1, u_2) := T_{k_1}(u_1 + 1) \cdots T_1(u_1 + k_1) \cdot T_{k_2+1}(u_2 + 1) \cdots T_2(u_2 + k_2)\psi_{12} \tag{A.12}$$

which we denote by

$\psi_{k_1+1,k_2+2}(u_1, u_2) =$

u_2+k_2						u_2+2	u_2+1
u_1+k_1			u_1+2	u_1+1			

.

Let

$$y_k = -\frac{[u_k]}{[u_k + 1]}.$$

Lemma 2. *For $a_2 > a_1 \geq 2$,*

$$\begin{aligned}\psi_{a_1a_2}(u_1, u_2) &= \Delta_t(z_1, \ldots, z_N) \oint \frac{\mathrm{d}w_1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}w_2}{2\pi\mathrm{i}}\ (w_2 - w_1)(tw_1 - t^{-1}w_2)\times \\ &\qquad (1 + y_1S_{a_1})\ (1 + y_2S_{a_2}) \cdot \phi_{a_1}(w_1)\phi_{a_2}(w_2) \\ &= \Delta_t(z_1, \ldots, z_N) \oint \frac{\mathrm{d}w_1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}w_2}{2\pi\mathrm{i}}\ (w_2 - w_1)(tw_1 - t^{-1}w_2)\times \\ &\qquad \prod_{m=1}^{2} \left(\frac{1}{[u_m + 1]} \frac{t^{u_m+1}w_m - t^{-u_m-1}z_{a_m}}{tw_m - t^{-1}z_{a_m}} \right) \phi_{a_m}(w_m).\end{aligned}$$

Proof. The proof again uses the induction argument of Lemma 1, as well as the fact that $T_a(u)$ commutes with functions symmetric in $z_1, \ldots, \hat{z}_a, \hat{z}_{a+1}, \ldots, z_N$. □

Example 20 ($N = 6$)**.** *Let*

$\psi_{2,4}(u_1, u_2) =$

u_2+2	u_2+1
u_1+1	

$= T_1(u_1 + 1)T_3(u_2 + 1)T_2(u_2 + 2)\Delta_t(z_1, z_2, z_3)\Delta_t(z_4, z_5, z_6).$

The integral representation for $\psi_{2,4}(u_1, u_2)$ is

$$\begin{aligned}\psi_{2,4}(u_1, u_2) &= \Delta_t(z_1, \ldots, z_N) \oint \frac{\mathrm{d}w_1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}w_2}{2\pi\mathrm{i}}\ (w_2 - w_1)(tw_1 - t^{-1}w_2)\times \\ &\qquad (1 + y_1S_2)\ (1 + y_2S_4) \cdot \phi_2(w_1)\phi_4(w_2). \\ &= \Delta_t(z_1, \ldots, z_N) \oint \frac{\mathrm{d}w_1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}w_2}{2\pi\mathrm{i}}\ (w_2 - w_1)(tw_1 - t^{-1}w_2)\times \\ &\qquad \left(\frac{1}{[u_1]} \frac{t^{u_1}w_1 - t^{-u_1}z_2}{tw_1 - t^{-1}z_2} \right) \phi_2(w_1) \left(\frac{1}{[u_2]} \frac{t^{u_2}w_2 - t^{-u_2}z_4}{tw_2 - t^{-1}z_4} \right) \phi_4(w_2).\end{aligned}$$

Appendix A.3. m rows

The proof of Theorem 1 can now be completed in similar fashion. Let us restate the result:

$$
\begin{aligned}
M_{[\lambda_n,\ldots,\lambda_1]} :=& \\
\psi_{a_1,\ldots a_n}(u_1,\ldots,u_n) = \Delta_t(z_1,\ldots,z_N) &\oint \frac{\mathrm{d}w_1}{2\pi\mathrm{i}} \cdots \oint \frac{\mathrm{d}w_n}{2\pi\mathrm{i}}\ \Delta(w_n,\ldots,w_1)\Delta_t(w_1,\ldots,w_n)\times \\
&\prod_{m=1}^{n} (1+y_m S_{a_m})\ \phi_{a_m}(w_m) \\
= \Delta_t(z_1,\ldots,z_N) &\oint \frac{\mathrm{d}w_1}{2\pi\mathrm{i}} \cdots \oint \frac{\mathrm{d}w_n}{2\pi\mathrm{i}}\ \Delta(w_n,\ldots,w_1)\Delta_t(w_1,\ldots,w_n)\times \\
&\prod_{m=1}^{n} \left(\frac{t^{u_m}w_m - t^{-u_m}z_{a_m}}{tw_m - t^{-1}z_{a_m}}\right)\phi_{a_m}(w_m).
\end{aligned}
$$

Proof. As before, by induction on $a_1,\ldots,a_n$, and the fact that $T_a(u)$ commutes with functions symmetric in $z_1,\ldots,\hat{z}_a,\hat{z}_{a+1},\ldots,z_N$. □

Appendix B. Proof of Proposition 1

The change of variables (16) results in

$$
\tilde{\Delta}_t(v_1,\ldots,v_N) = \Delta_t(z_1,\ldots,z_N) = \frac{(t-t^{-1})^{N(N-1)/2}}{\prod_{i=1}^{N}(1-tv_i)^{N-1}} \prod_{1\le i<j\le N}(1+v_iv_j+\tau v_j)
$$

and

$$
\phi_i(x) = \prod_{j=1}^{N} \frac{(1-tv_j)(1-tx)}{t-t^{-1}} \prod_{j=1}^{i} \frac{1}{x-v_j} \prod_{j=i+1}^{N} \frac{1}{1+xv_j+\tau v_j}\ .
$$

In the homogeneous limit, $v_j \to 0$, this reduces to

$$
\phi_i(x) \to \left(\frac{1-tx}{t-t^{-1}}\right)^N \frac{1}{x^i}\ .
$$

We thus find that $\psi_{a_1,\ldots a_n}(u_1,\ldots,u_n)$ can be written as

$$
\begin{aligned}
\psi_{a_1,\ldots a_n}(u_1,\ldots,u_n) = C_{N,n} &\prod_{1\le i<j\le N}(1+v_iv_j+\tau v_j)\prod_{j=1}^{N}(1-tv_j)^{n-N+1}\times \\
&\oint \frac{\mathrm{d}x_1}{2\pi\mathrm{i}} \cdots \oint \frac{\mathrm{d}x_k}{2\pi\mathrm{i}} \prod_{1\le i<j\le n}(x_j-x_i)(1+x_ix_j+\tau x_j)\prod_{m=1}^{n}(1-tx_m)^{N-2n} \\
&\prod_{m=1}^{n}\left(\frac{1+x_mv_{a_m}-x_m\frac{[u_m]}{[u_m+1]}-v_{a_m}\frac{[u_m+2]}{[u_m+1]}}{1+x_mv_{a_m}+\tau v_{a_m}}\prod_{i=1}^{a_m}\frac{1}{x_m-v_i}\prod_{i=a_m+1}^{N}\frac{1}{1+x_mv_i+\tau v_i}\right)\ ,
\end{aligned}
$$

where the integration is around the points v_i. The homogenous limit $v_i \to 0$ can now easily be taken.

Appendix C. Expansion of $\mathrm{M}_{[3,2,1]}$ in terms of KL polynomials

We consider $n = 3$. We write the expansion (13) of the deformed maximal Macdonald polynomial $\mathrm{M}_{[2,1,0]}(u_2, u_3)$ as,

$$\begin{array}{|c|c|}\hline 2+u_3 & 1+u_3 \\ \hline 1+u_2 & \\ \hline\end{array} = c_{[2,1,0]}\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \hline\end{array} + c_{[1,1,0]}\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} + c_{[2,0,0]}\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} + c_{[1,0,0]}\begin{array}{|c|}\hline 1 \\ \hline\end{array} + c_{[0,0,0]}\emptyset.$$

Now, using (14) we can write the expansion for $\mathrm{M}_{[3,2,0]}$,

$$M_{[3,2,0]}(u_2, u_3) = \begin{array}{|c|c|c|}\hline 3+u_3 & 2+u_3 & 1+u_3 \\ \hline 2+u_2 & 1+u_2 & \\ \hline\end{array} = c_{[2,1,0]}\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 2 & 1 & \\ \hline\end{array} + c_{[1,1,0]}\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline\end{array} +$$

$$c_{[2,0,0]}\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 & & \\ \hline\end{array} + c_{[1,0,0]}\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \hline\end{array} + c_{[0,0,0]}\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}. \quad \text{(C.1)}$$

We want to obtain the expansion for $\mathrm{M}_{[3,2,1]}(u_1, u_2, u_3)$, and so by acting with $T_2(u_1+1)$ we obtain

$$M_{[3,2,1]}(u_1, u_2, u_3) = T_2(u_1 + 1)M_{[3,2,0]}(u_2, u_3) = (T_2(1) + y_1)M_{[3,2,0]}(u_2, u_3)$$

$$= c_{[2,1,0]}\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 2 & 1 & \\ \hline 1 & & \\ \hline\end{array} + c_{[1,1,0]}\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline 1 & \\ \hline\end{array} + c_{[2,0,0]}\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 & & \\ \hline 1 & & \\ \hline\end{array} + c_{[1,0,0]}\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \hline 1 & \\ \hline\end{array} + c_{[0,0,0]}\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 1 \\ \hline\end{array}$$

$$+y_1\left(c_{[2,1,0]}\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 2 & 1 & \\ \hline\end{array} + c_{[1,1,0]}\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline\end{array} + c_{[2,0,0]}\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 & & \\ \hline\end{array} + c_{[1,0,0]}\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \hline\end{array} + c_{[0,0,0]}\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}\right). \quad \text{(C.2)}$$

Now, the third, fourth and fifth elements in this expansion are <u>not</u> KL elements; however, they may be expanded in terms of KL elements, for example we can write

$$\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 & & \\ \hline 1 & & \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 2 & & \\ \hline 1 & & \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline\end{array}. \quad \text{(C.3)}$$

Now we see how this result is obtained using the expansion over Dyck ribbons in (15), which we recall here:

$$T_2(u_1 + 1)\ \mathrm{KL}_{[\lambda]} = \mathrm{KL}_{[\lambda,1]} + y_1\mathrm{KL}_{[\lambda]} + \sum_{\mu\in R_\rho} \mathrm{KL}_{[\lambda/\mu]}. \quad \text{(C.4)}$$

If we consider again the third element on the right hand side of (C.2), we note that it arises from the third element in the right hand side of (C.1), i.e. the partition $\lambda = [3, 1, 0]$. We thus need to consider all Dyck ribbons with the diagonal as shown:

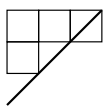


Figure C.12: The diagonal corresponding to $\lambda = [3, 1, 0]$

There is only one such Dyck ribbon, shown shaded in Figure C.13. So the only partition $[\lambda/\mu]$ appearing in the sumon the right hand side of (C.4) is $[3, 0, 0]$. Thus, from (C.4), the action of $T_2(u_1 + 1)$ on $\mathrm{KL}_{[3,1,0]}$ is

$$T_2(u_1 + 1)\ \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 & & \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 2 & & \\ \hline 1 & & \\ \hline\end{array} + \ y_1\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 & & \\ \hline\end{array} + \ \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline\end{array},$$

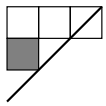


Figure C.13: The Dyck ribbon

in agreement with the result from the expansion (C.2), using the result (C.3).

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